A New Infeasible Interior-Point Algorithm for Linear Programming *

Miguel Argáez
mar@math.utep.edu

Leticia Velázquez
leti@math.utep.edu

Department of Mathematical Sciences
The University of Texas at El Paso
El Paso, Texas 79968-0514

ABSTRACT

In this paper we present an infeasible path-following interior-point algorithm for solving linear programs using a relaxed notion of the central path, called quasicentral path, as a central region. The algorithm starts from an infeasible point which satisfies that the norm of the dual condition is less than the norm of the primal condition. We use weighted sets as proximity measures of the quasicentral path, and a new merit function for making progress toward this central region. We test the algorithm on a set of NETLIB problems obtaining promising numerical results.

Categories and Subject Descriptors

G.4 [**Mathematics of Computing**]: Mathematical Software—*efficiency*; G.1.6 [**Numerical Analysis**]: Optimization—*linear programming*

General Terms

Algorithms, Experimentation

Keywords

Interior-point methods, primal-dual, Newton's method, merit function

1. INTRODUCTION

We consider the primal linear problem in the standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array} \tag{1}$$

where c, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{mxn}$, $m < n$, and full rank. The dual linear problem associated with problem (1) can be written

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + z = c \\ & z \geq 0, \end{array} \tag{2}$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.
A point (x, y, z) is said to be an interior point for problems (1) and (2) if $(x, z) > 0$.
For $\mu > 0$, a point on the central path satisfies the perturbed Karush-Kuhn-Tucker (KKT) conditions associated with problems (1) and (2) are

$$F_\mu(x, y, z) = \begin{pmatrix} Ax - b \\ A^T y + z - c \\ XZe - \mu e \end{pmatrix} = 0,$$

$$(x, z) \geq 0,$$

where $X = \text{diag}(x),\ Z = \text{diag}(z)$, and $e = (1, \ldots, 1)^T \in \mathbb{R}^n$. The Newton direction for this system is

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} b - Ax \\ c - A^T y - z \\ \mu e - XZe \end{pmatrix}. \tag{3}$$

Now, the perturbed KKT conditions promote global convergence and avoids the procedure for converging to spurious solutions. The central path has the property that runs through the strictly feasible set

$$\mathcal{F}^o = \{(x, y, z) \in \mathbb{R}^{2n+m} : Ax = b,\ A^T y + z = c,\ (x, z) > 0\},$$

keeps iterates at an adequate distance from the non-optimal borders, and ends at a particular solution called the analytic center.

Due to this property, most of the primal-dual interior-point algorithms are based on following explicitly or implicitly the central path as a guide for obtaining a solution of the primal and dual problems. Contrary to this central region, we use a relaxed notion of the central path called quasicentral path introduced in nonlinear programming by [1, 2], and analyzed in [3]. The use of the quasicentral path, as opposed to the central path, yields a definite advantage. Specifically, for problems where the strictly feasible set $\mathcal{F}^o$ is empty.

*This work was supported by the US Department of the Army under Grant DAAD19-01-1-0741.

2. PATH-FOLLOWING STRATEGY

We present a general description of a path-following strategy as follows: For a fixed $\mu > 0$, working from an interior point (x, y, z), apply a linesearch Newton's method to the perturbed KKT conditions until an iterate arrives at a specified measure of nearness of the central region. If an optimal solution is not found, decrease μ and the process is repeated. A solution is found as μ approaches zero.

The quasicentral path, introduced by [1, 2], is defined as the collection of points $(x, z) > 0$ parameterized by $\mu > 0$ such that

$$\begin{aligned} Ax - b &= 0, \quad \text{and} \\ XZe - \mu e &= 0\,. \end{aligned}$$

The quasicentral path is equivalent to the region of strictly feasible points for the primal problem.

Now, let us define the errors for the primal and dual equations as

$$e_p^k = b - Ax_k, \quad \text{and} \quad e_d^k = c - A^T y_k - z_k.$$

PROPOSITION 1. *If the initial point is such that $\|e_d^0\| \le \|e_p^0\|$, then the dual error, e_d^k, is zero if the primal error, e_p^k, is zero.*

Proof. By applying a damped Newton's method to primal and dual residuals at an infeasible starting point, then

$$\begin{aligned} e_p^1 &= b - Ax_1 = (1-\alpha_1)\, e_p^o, \quad \text{and} \\ e_d^1 &= c - A^T y_1 - z_1 = (1-\alpha_1)\, e_d^o. \end{aligned}$$

Iteratively, we obtain

$$e_p^k = b - Ax_k = \prod_{j=1}^{k}(1-\alpha_j)\, e_p^o, \quad \text{and}$$

$$e_d^k = c - A^T y_k - z_k = \prod_{j=1}^{k}(1-\alpha_j)\, e_d^o.$$

The proof follows from the last two equations, and the fact that $\|e_d^0\| \le \|e_p^0\|$. □

This property suggests that the quasicentral path can be used as a central region to reach a solution of the problem. Instead of following the quasicentral path exactly, which requires high computational cost, proximity measures of the quasicentral path are defined. For $\mu > 0$ and $\gamma \in (0, 1)$, we say that a point $(x > 0, z > 0) \in \mathbb{R}^{n+n}$ is sufficiently close to the quasicentral path, if it is contained in the following neighborhood:

$$\mathcal{N}(\gamma\mu) = \{\|Ax - b\|^2 + \|(XZ)^{-1/2}(XZe - \mu e)\|^2 \le \gamma\mu\}.$$

To make progress to the quasicentral path, we use the following function as a merit function:

$$\Phi_\mu(x, z) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{n}(x_i z_i - \mu \ln(x_i z_i)).$$

The Newton step is a descent direction for this function at any point that is not in the quasicentral path.

3. INFEASIBLE ALGORITHM

We present an infeasible path-following algorithm for computing a solution that uses the primal-dual interior-point framework proposed by [4]. The quasicentral path is used as a central region. As a measure of nearness to the quasicentral path and to make progress towards this region, we use the set $\mathcal{N}(\gamma\mu)$ and merit function $\Phi_\mu(x, z)$, respectively.

Table 1: Numerical Results for Algorithm 3.1

Problem	m	n	No. of Iterations	$\|e_d^0\|$	$\|e_p^0\|$
afiro	27	51	9	3.07e1	2.94e3
blend	74	114	16	4.00e1	2.06e3
adlittle	55	137	18	1.93e4	2.80e4
sc205	205	317	15	4.46e1	2.38e3
sc50a	49	77	11	2.21e1	1.13e3
sc50b	48	76	10	2.19e1	1.47e3
scsd1	77	760	12	8.12e2	8.91e2
scsd6	147	1350	15	1.38e3	1.51e3
scagr7	129	185	15	1.43e4	2.00e4
sctap1	300	660	23	4.86e3	1.51e6

Algorithm 3.1

(1) Given an interior point (x, z), ϵ, $\mu > 0$, and $\tau, \gamma \in (0, 1)$.
(2) Initialize $e_d = c - z$, $e_p = b - Ax$, and $e_c = \mu e - XZe$ where $\|e_d\| \le \|e_p\|$.
(3) Repeat Steps (3)(a)-(3)(f) until step (3)(f) is satisfied.
 (a) Newton direction. Solve system (3) for $(\Delta x, \Delta y, \Delta z)$.
 (b) Maintain x and z positive. Calculate $\tilde{\alpha} = \min(1, \tau\hat{\alpha})$ where $\hat{\alpha} = \frac{-1}{\min(X^{-1}\Delta x, Z^{-1}\Delta z)}$.
 (c) Sufficient decrease. Find $\alpha = (\frac{1}{2})^t \tilde{\alpha}$ where t is the smallest positive integer such that
 $\Phi_\mu(x + \alpha\Delta x, z + \alpha\Delta z) \le \Phi_\mu(x, z) + 10^{-4}\alpha *$
 $\nabla\Phi_\mu(x, z)^T(\Delta x, \Delta z)$.
 (d) Update $(x, z) = (x, z) + \alpha(\Delta x, \Delta z)$, $e_d = (1-\alpha)e_d$, and $e_p = (1-\alpha)e_p$.
 (e) Proximity to the quasicentral path.
  if ($\|e_p\|^2 + \|(XZ)^{-1/2}(XZe - \mu e)\|^2) \le \gamma\mu$,
   then go to step (3)(f).
  else set $e_c = \mu e - XZe$,
   and go to step (3)(a).
 (f) Stopping criteria.
  if $\left(\frac{2\|e_p\|}{\max(1,\|b\|,\|c\|)} + \frac{x^T z}{\max(1,|c^T x|)}\right) \le \epsilon$, STOP.
  else update μ, set $e_c = \mu e - XZe$,
   and go to step (3)(a).
(4) Termination. Return x.

4. NUMERICAL EXPERIMENTATION

The Algorithm 3.1, written in MATLAB, was run on a set of ten NETLIB problems using a Sun Ultra 10 machine. We report the number of Newton iterations it took for solving each problem on Table I. The perturbation parameter μ is updated by $\mu = 10^{-2} * (\ \|e_p\|^2 + \|(XZ)^{-1/2}(XZe - \mu e)\|^2)$ when the iterate (x, z) is inside the proximity measure set. We set the initial value of $y = 0$, and choose the initial point (x, z) using a standard procedure. Now if $\|b - Ax\| > \|c - z\|$, we find a positive scalar ζ such that $\|c - z\| \le \|b - A(\zeta x)\|$, and let the initial point be $(\zeta x, z)$.

5. CONCLUSIONS

We present an infeasible primal-dual interior-point algorithm for linear programming featuring a new central region called quasicentral path, a new merit function, and weighted proximity measures. The dual variable y does not play any role, at least explicity, in our procedure. The numerical results indicate the quasicentral path is enough to guide the iterates to a solution of the problem. Further numerical and theoretical research is needed to establish the role that the quasicentral path plays for solving linear programming problems.

6. ACKNOWLEDGMENTS

The authors thank the referees for their valuable comments.

7. REFERENCES

[1] M. Argáez. *Exact and Inexact Newton Linesearch Interior-Point Algorithms for Nonlinear Programming Problems*. PhD thesis, Rice University, Houston, TX, 1997.

[2] M. Argáez and R. A. Tapia. On the global convergence of a modified augmented lagrangian linesearch interior-point newton method for nonlinear programming. *J. Optim. Theor. Appl.*, 114(1):1–25, August 2002.

[3] M. Argáez, R. A. Tapia, and L. Velázquez. Numerical comparisons of path–following strategies for a primal–dual interior–point method for nonlinear programming. *J. Optim. Theor. Appl.*, 114(2):255–272, August 2002.

[4] M. Kojima, S. Mizuno, and A. Yoshise. A primal-dual interior point method for linear programming. *Progress in Mathematical Programming: Interior-Point and Related Methods*, pages 29–47, 1989.